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The Metric Projection for Free Knot Splines

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Only a few results are known on continuity properties of the set-valued metric projection in nonlinear uniform approximation. In this paper we investigate this mapping in the case of best uniform approximation by splines of degree m with k free knots. A characterization of those functions at which the metric projection is upper semicontinuous is given. It is found that the metric projection is upper semicontinuous if and only if $k \leqslant m$, and that it is upper semicontinuous at all "normal" functions. On the other hand, it is shown that the metric projection is never lower semicontinuous. © 1992 Academic Press, Inc.

1. INTRODUCTION

There is a vast literature on continuity properties of the set-valued metric projection onto linear subspaces (see, e.g., the surveys Deutsch [7, 8], Nürnberger and Sommer [14], Singer [18], Vlasov [19], and the references therein). On the other hand, not as many results are known about this mapping in nonlinear approximation (see, e.g., Berens and Finzel [1], Brosowski and Deutsch [5], Deutsch [6], Nürnberger [9], Schmidt [15], and Singer [18]).

The aim of this paper is to investigate the metric projection onto $S_{m,k}$, the set of polynomial splines of degree m with k free knots. This is the mapping which associates to each function $f \in C[a, b]$ the set $P_{S_{m,k}}(f) = \{s_f \in S_{m,k} : \|f - s_f\| = \inf_{s \in S_{m,k}} \|f - s\|\}$ of its best uniform approximations from $S_{m,k}$. We give a characterization of those functions in $C[a, b]$ at which $P_{S_{m,k}}$ is upper semicontinuous. As a consequence we get that $P_{S_{m,k}}$ is upper semicontinuous on $C[a, b]$ if and only if $k \leqslant m$. Moreover, it follows that $P_{S_{m,k}}$ is upper semicontinuous on the set $\{f \in C[a, b] : P_{S_{m,k}}(f) \subset C[a, b]$ and $P_{S_{m,k}}(f) \cap S_{m,k-1} = \varnothing\}$. On the other hand, we show that $P_{S_{m,k}}$ is never lower semicontinuous.

The same statements hold for the set-valued mapping which associates to each function $f \in [a, b]$, the nonempty set $P_{S_{m,k}}(f) \cap C[a, b]$ of its continuous best approximations.

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In a further paper we apply the results to derive uniqueness theorems (announced in [12]) for $S_{m,k}$.

2. Main Results

Let $C[a, b]$ be the space of all continuous real-valued functions f on $[a, b]$ endowed with the supremum norm $\|f\| = \sup_{t \in [a,b]} |f(t)|$. Moreover, let points $a = x_0 < x_1 < \cdots < x_r < x_{r+1} = b$ and integers $m_1, ..., m_r \in \{1, ..., m+1\}$ be given, where $m \geq 1$ and $r \geq 1$. We denote by $S_m\binom{x_1, ..., x_r}{m_1, ..., m_r}$ the space of polynomial splines of degree m with r fixed knots $x_1, ..., x_r$ of multiplicities $m_1, ..., m_r$, and by $S_{m,k}$ the set of polynomial splines of degree m with k free (multiple) knots, where $k \geq 1$ (see, e.g., Nürnberger [11] and Schumaker [17]). Here we use the convention that a spline has a knot of multiplicity $m+1$ if for this spline no continuity is required at the knot.

A spline $s_f \in S_{m,k}$ is called the *best uniform approximation* of a function $f \in C[a, b]$ from $S_{m,k}$ if $\|f - s_f\| = \inf_{s \in S_{m,k}} \|f - s\|$. The nonempty set of best uniform approximations of f from $S_{m,k}$ is denoted by $P_{S_{m,k}}(f)$, and the resulting set-valued mapping $P_{S_{m,k}}: C[a, b] \to 2^{S_{m,k}}$ is called the *metric projection* onto $S_{m,k}$.

In the following we investigate continuity properties of this mapping.

Definition 1. The metric projection $P_{S_{m,k}}: C[a, b] \to 2^{S_{m,k}}$ is called *upper semicontinuous* (u.s.c.) (respectively *lower semicontinuous* (l.s.c.)) at $f \in C[a, b]$ if for each sequence $(f_n) \subset C[a, b]$ with $f_n \to f$ and each closed subset A of $S_{m,k}$ with $P_{S_{m,k}}(f_n) \cap A \neq \emptyset$ (respectively $P_{S_{m,k}}(f_n) \subset A$) for all n, we have $P_{S_{m,k}}(f) \cap A \neq \emptyset$ (respectively $P_{S_{m,k}}(f) \subset A$). $P_{S_{m,k}}$ is called *upper semicontinuous* (respectively *lower semicontinuous*) if it is u.s.c. (respectively l.s.c.) at every function $f \in C[a, b]$.

The first result shows that the upper semicontinuity of the metric projection $P_{S_{m,k}}$ at a given function depends on the multiplicities of the knots of its best approximations from $S_{m,k}$.

Theorem 1. *For a function $f \in C[a, b] \backslash S_{m,k}$, the following statements are equivalent*:

(i) *$P_{S_{m,k}}$ is upper semicontinuous at f.*

(ii) *There does not exist a spline $s \in P_{S_{m,k}}(f) \cap S_m\binom{x_1, ..., x_r}{m_1, ..., m_r}$ such that s is discontinuous or $m + 2 + \sum_{i=1}^r m_i - \max_{i=1,...,r} m_i \leq k$.*

Proof. (ii) $\to$ (i). Suppose that (ii) holds. Let a closed set A in $S_{m,k}$, $f \in C[a, b]$ and $(f_n) \subset C[a, b]$ be given such that $f_n \to f$ and $P_{S_{m,k}}(f_n) \cap$

$A \neq \emptyset$ for all n. We have to show that $P_{S_{m,k}}(f) \cap A \neq \emptyset$ which implies that $P_{S_{m,k}}$ is upper semicontinuous at f. For all n, we choose a spline $s_n \in P_{S_{m,k}}(f_n) \cap A$. We will show that there exist a spline $s \in P_{S_{m,k}}(f)$ and a subsequence (s_{n_q}) of (s_n) such that $\lim_{q \to \infty} \|s - s_{n_q}\| = 0$. Since A is closed, it follows that $s \in A$ which proves the claim. It is easy to see that (s_n) is a bounded sequence. Therefore, it follows from Braess [4, p. 229] that there exists a spline $s \in P_{S_{m,k}}(f) \cap S_m\binom{x_1, \ldots, x_r}{m_1, \ldots, m_r}$ such that a subsequence of (s_n), again denoted by (s_n), converges to s uniformly on each compact subset of $[a, b] \setminus \{x_1 \ldots, x_r\}$. Moreover, the knots of (s_n) converge to the knots of s. It follows from (ii) that s is continuous and $m + 2 + \sum_{i=1}^{r} m_i - \max_{i=1,\ldots,r} m_i > k$. For all $i \in \{1, \ldots, r\}$, let m_i be the minimal multiplicity of x_i such that $s \in S_m\binom{x_1, \ldots, x_r}{m_1, \ldots, m_r}$. Now, let an index $j \in \{1, \ldots, r\}$ be given. By going to a subsequence, we may assume that for all n, the same number of (multiple) knots of s_n, say $y_{1,n} \leqslant \cdots \leqslant y_{p_j,n}$, converges to x_j. Then we have $p_j \geqslant m_j$, because, if $p_j < m_j \leqslant m$, then it follows from Braess [4, p. 229] that

$$\|s - s_n\|_{[(1/2)(x_{j-1} + x_j),\, (1/2)(x_j + x_{j+1})]} \to 0$$

and that s has a knot of multiplicity p_j at x_j which is a contradiction. Moreover, we have $p_j \leqslant m + 1$, because, if $p_j \geqslant m + 2$, then, since (ii) holds,

$$\sum_{i=1}^{r} p_i \geqslant m + 2 + \sum_{\substack{i=1 \\ i \neq j}}^{r} m_i \geqslant m + 2 + \sum_{i=1}^{r} m_i - \max_{i=1,\ldots,r} m_i > k$$

which is a contradiction to $s_n \in S_{m,k}$. We define

$$K_m(z, t) = (t - z)_+^m, \qquad (z, t) \in [a, b] \times [a, b]$$

and denote by $K_m[z_1, \ldots, z_{l+1}, t]$ the divided difference of order l of the function $z \to K_m(z, t)$ with respect to the points $z_1, \ldots, z_{l+1}$. Then for all n, the spline s_n can be written as

$$s_n(t) = \sum_{i=0}^{m} a_{i,n} t^i + \sum_{i=1}^{p_j} b_{i,n} K_m[y_{1,n}, \ldots, y_{i,n}, t],$$

$$t \in \left[\frac{1}{2}(x_{j-1} + x_j), \frac{1}{2}(x_j + x_{j+1})\right].$$

For sufficiently large n, we have

$$x_{j-1} + \tfrac{3}{4}(x_j - x_{j-1}) \leqslant y_{1,n} \leqslant \cdots \leqslant y_{p_j,n} \leqslant x_j + \tfrac{1}{4}(x_{j+1} - x_j).$$

Now, we choose points $t_1, ..., t_{m+p_j+1}$ such that

$$\tfrac{1}{2}(x_{j-1}+x_j) \leqslant t_1 < \cdots < t_{m+1} < x_{j-1} + \tfrac{3}{4}(x_j - x_{j-1}) < x_j + \tfrac{1}{4}(x_{j+1} - x_j)$$
$$< t_{m+2} < \cdots < t_{m+p_j+1} \leqslant \tfrac{1}{2}(x_j + x_{j+1}).$$

It is well known and easy to verify that the determinant generated by inserting these points into the $m+p_j+1$ functions

$$1, t, ..., t^m, K_m[x_j, \cdot], ..., K_m[x_j, ..., x_j, \cdot]$$

is different from zero. Therefore, since (s_n) is bounded and for all $t \in [a, b] \backslash \{x_j\}$,

$$K_m[y_{1,n}, ..., y_{i,n}, t] \to K_m[x_j, ..., x_j, t], \qquad i = 1, ..., p_j,$$

the sequences $(a_{i,n})$, $i = 0, ..., m$, and $(b_{i,n})$, $i = 1, ..., p_j$, are bounded. Thus by going to subsequences, we may assume that these sequences converge.

Moreover, since the spline s is continuous, we have $\lim_{n\to\infty} b_{m+1,n} = 0$, if $p_j = m+1$. This implies that

$$\|s - s_n\|_{[(1/2)(x_{j-1}+x_j),\,(1/2)(x_j+x_{j+1})]} \to 0.$$

Since this holds for every index $j \in \{1, ..., r\}$, it follows that $\|s - s_n\| \to 0$.

(i) $\to$ (ii). Suppose that (ii) fails. We will show that $P_{S_{m,k}}$ is not upper semicontinuous at f. We first assume that there exists a spline $s \in P_{S_{m,k}}(f)$ which is discontinuous at some knot x_j. Then it follows from Schumaker [16] (see also Braess [4, p. 230]) that there exists a sequence $(\tilde{s}_n) \subset P_{S_{m,k}}(f)$ with the following properties. For all n, the spline $\tilde{s}_n$ has a simple knot at $x_j - \alpha_n$ and a knot of multiplicity m at $x_j + \beta_n$, where $\alpha_n > 0$, $\beta_n > 0$ and $\alpha_n \to 0$, $\beta_n \to 0$.

Moreover, for all n,

$$\tilde{s}_n(t) = s(t), \qquad t \in [a, b] \backslash (x_j - \alpha_n, x_j + \beta_n),$$

and

$$\tilde{s}_n(t) \to s(t), \qquad t \in [a, b] \backslash \{x_j\}.$$

We set for all n, $s_n = \tilde{s}_n + 1/n$ and $f_n = f + 1/n$. Since $f - s$ has alternating extreme points, for all n, $s_n \notin P_{S_{m,k}}(f)$. Moreover, since $\tilde{s}_n \in P_{S_{m,k}}(f)$, it follows that $s_n \in P_{S_{m,k}}(f_n)$. The set $A = \{s_n : n \in \mathbb{N}\}$ is closed, since no subsequence of (s_n) converges uniformly. Now, since $f_n \to f$, $P_{S_{m,k}}(f_n) \cap A \neq \emptyset$ for all n, but $P_{S_{m,k}}(f) \cap A = \emptyset$, the metric projection $P_{S_{m,k}}$ is not upper semicontinuous at f.

Finally, suppose that there exists a spline

$$s \in P_{S_{m,k}}(f) \cap S_m \begin{pmatrix} x_1, ..., x_r \\ m_1, ..., m_r \end{pmatrix} \subset C[a, b]$$

such that $m + 2 + \sum_{i=1}^{r} m_i - \max_{i=1,...,r} m_i \leqslant k$. Let x_j be a knot with $m_j = \max_{i=1,...,r} m_i \leqslant m$. We set

$$y_{i,n} = x_j, \qquad i = 2, ..., m_j + 1,$$

and choose points

$$y_{1,n} < x_j < y_{m_j+2,n} < \cdots < y_{m+2,n}$$

such that

$$y_{i,n} \to x_i, \qquad i = 1, ..., m+2.$$

Let B_n be the normalized B-spline of degree m associated with the knots

$$y_{1,n} \leqslant \cdots \leqslant y_{m+2,n}.$$

By multiplying B_n by an appropriate factor for all n, we may assume that

$$B_n(x_j) = \tfrac{1}{2}(f(x_j) - s(x_j)) \qquad \text{if} \quad f(x_j) - s(x_j) \neq 0$$

and

$$B_n(x_j) = \tfrac{1}{2} \|f - s\| \qquad \text{if} \quad f(x_j) - s(x_j) = 0.$$

For all n, we set $\tilde{s}_n = s + B_n$. Then for sufficiently large n, $\tilde{s}_n \in P_{S_{m,k}}(f)$. As above, we set, for all n, $s_n = \tilde{s}_n + 1/n$, $f_n = f + 1/n$, and $A = \{s_n : n \in \mathbb{N}\}$. Since no subsequence of (s_n) converges uniformly, the set A is closed. Analogously as above, we have $f_n \to f$ and $P_{S_{m,k}}(f_n) \cap A \neq \emptyset$ for all n, but $P_{S_{m,k}}(f) \cap A = \emptyset$. Therefore, $P_{S_{m,k}}$ is not upper semicontinuous at f. This proves Theorem 1.

The proof of Theorem 1 gives the following result on the convergence of sequences in $S_{m,k}$.

PROPOSITION. *For a spline* $s \in S_{m,k} \cap S_m\binom{x_1, ..., x_r}{m_1, ..., m_r}$, *the following statements are equivalent*:

(i) *If a sequence* (s_n) *in* $S_{m,k}$ *converges pointwise to* s *(except at the knots* $x_1, ..., x_r$*), then* s_n *converges uniformly to* s.

(ii) s *is continuous and* $m + 2 + \sum_{i=1}^{r} m_i - \max_{i=1,...,r} m_i > k$.

As a first direct consequence of Theorem 1, we obtain a characterization of the upper semicontinuity of $P_{S_{m,k}}$.

COROLLARY 1. *The metric projection $P_{S_{m,k}}$ is upper semicontinuous on $C[a, b]$ if and only if $k \leqslant m$.*

Proof. It is easy to verify that $P_{S_{m,k}}$ is upper semicontinuous on $S_{m,k}$. Suppose that $k \leqslant m$ and let $f \in C[a, b] \backslash S_{m,k}$ be given. Then all splines $s \in P_{S_{m,k}}(f)$ are continuous and the inequality in Theorem 1 is obviously not satisfied for s. Therefore, it follows from Theorem 1 that $P_{S_{m,k}}$ is upper semicontinuous at f.

Now, suppose that $k > m$. Then there exists a spline $s \in S_{m,k}$ which is not continuous. It is clear that we can construct a function $f \in C[a, b] \backslash S_{m,k}$ such that $f - s$ has $m + 2k + 2$ alternating extreme points on some knot-interval of s. Then by Schumaker [16], $s \in P_{S_{m,k}}(f)$ and by Theorem 1, $P_{S_{m,k}}$ is not upper semicontinuous. This proves Corollary 1.

The second conclusion of Theorem 1 shows that $P_{S_{m,k}}$ is upper semicontinuous on a large subset of $C[a, b]$, namely at all "normal" functions.

COROLLARY 2. *The metric projection $P_{S_{m,k}}$ is upper semicontinuous on*

$$\{f \in C[a, b] : P_{S_{m,k}}(f) \subset C[a, b] \text{ and } P_{S_{m,k}}(f) \cap S_{m,k-1} = \varnothing\}.$$

Proof. Let a function $f \in C[a, b]$ be given such that $P_{S_{m,k}}(f) \subset C[a, b]$ and $P_{S_{m,k}}(f) \cap S_{m,k-1} = \varnothing$. This means that for all $s \in P_{S_{m,k}}(f) \cap S_m\binom{x_1, \ldots, x_r}{m_1, \ldots, m_r}$, we have $m_i \leqslant m$, $i = 1, \ldots, r$, and $\sum_{i=1}^{r} m_i = k$. Therefore, the inequality in Theorem 1 cannot be satisfied and $P_{S_{m,k}}$ is upper semicontinuous at f. This proves Corollary 2.

While by Corollary 1, the metric projection $P_{S_{m,k}}$ is upper semicontinuous if and only if $k \leqslant m$, we now show that $P_{S_{m,k}}$ is never lower semicontinuous.

THEOREM 2. *The metric projection $P_{S_{m,k}} : C[a, b] \to 2^{S_{m,k}}$ is not lower semicontinuous.*

Proof. We construct a function $f \in C[a, b]$ and a sequence (f_n) in $C[a, b]$ such that $f_n \to f$, $P_{S_{m,k}}(f_n) = \{s_0\}$ for all n and $\{s_0\} \subsetneqq P_{S_{m,k}}(f)$, which shows that $P_{S_{m,k}}$ is not lower semicontinuous. To do this, we choose arbitrary points

$$a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$$

and a spline $s_0 \in S_{m,k} \backslash S_{m,k-1}$ which has active knots at $x_1, ..., x_k$ such that $s_0(t) = (t - x_k)^m$, $t \in [x_{k-1}, x_k]$, and $s_0(t) = 0$, $t \in [x_k, b]$. Moreover, we define $f \in C[x_k, x_{k+1}]$ such that $f(x_k) = -1$, $f((x_k + x_{k+1})/2) = 1$, $f(x_{k+1}) = -1$ and f is linear elsewhere on $[x_k, x_{k+1}]$. We may extend f to a function in $C[a, b]$ such that $\|f - s_0\| = 1$, $f - s_0$ is piecewise linear and $f - s_0$ has sufficiently many (which will be specified later) alternating extreme points on each knot-interval $[x_i, x_{i+1}]$, $i = 0, ..., k-1$. We now define a sequence (f_n) in $C[a, b]$ as follows. For all n, we set

$$f_n(t) = f(t), \qquad t \in [a, x_k] \cup [x_k + 1/n, b],$$

$$f_n(t) = -1, \qquad t \in [x_k, x_k + 1/2n],$$

$$f_n \text{ is linear on } (x_k + 1/2n, x_k + 1/n).$$

Then it follows that $f_n \to f$.

Now, let $y_1 \leqslant \cdots \leqslant y_{2k}$ be the knots of s_0 counting each knot twice. Moreover, we choose arbitrary points $y_{-m} < \cdots < y_{-1} < y_0 = a$ and $b = y_{2k+1} < y_{2k+2} < \cdots < y_{2k+m+1}$. We have the freedom to define f on $[a, x_k]$ such that for all n, $f_n - s_0$ has at least $j+1$ alternating extreme points in each knot-interval $(y_i, y_{i+m+j}) \subset (y_{-m}, y_{2k+m+1})$, $j \geqslant 1$.

Note that by construction the interval $(y_{2k-1}, y_{2k+m+1}) \subset (y_{-m}, y_{2k+m+1})$, $j \geqslant 1$, contains three alternating extreme points of $f_n - s_0$ for all n, but only two alternating extreme points of $f - s_0$.

Moreover, by construction $f - s_0$ has the same number of alternating extreme points on $[a, b]$ as $f - s_n$, and therefore, $f - s_0$ has at least $m + 2k + 2$ alternating extreme points on (y_{-m}, y_{2k+m+1}). Therefore, it follows from Schumaker [16] and Braess [3] that $s_0 \in P_{S_{m,k}}(f)$. Moreover, since $f_n - s_0$ has sufficiently many alternating extreme points in each interval (y_i, y_{i+m+j}), it follows from Nürnberger [10] that s_0 is a (strongly) unique best approximation of f_n from $S_{m,k}$ for all n. We now show that $\{s_0\} \neq P_{S_{m,k}}(f)$. For all $\varepsilon > 0$ we define $s_\varepsilon \in S_{m,k} \backslash S_{m,k-1}$ by

$$s_\varepsilon(t) = s_0(t), \qquad t \in [a, x_{k-1}],$$

$$s_\varepsilon(t) = (t - x_k)^m, \qquad t \in [x_{k-1}, x_k + \varepsilon],$$

and

$$s_\varepsilon(t) = (t - x_k)^m + \alpha_\varepsilon(t - (x_k + \varepsilon))^m, \qquad t \in [x_k + \varepsilon, b],$$

where

$$\alpha_\varepsilon = -(\tfrac{3}{4}(x_{k+1} - x_k))^m / (\tfrac{3}{4}(x_{k+1} - x_k) - \varepsilon)^m.$$

Then it follows that

$$s_\varepsilon(t) > 0, \qquad t \in (x_k, x_k + \tfrac{3}{4}(x_{k+1} - x_k)),$$

and

$$s_\varepsilon(t) < 0, \qquad t \in (x_k + \tfrac{3}{4}(x_{k+1} - x_k), b].$$

Since f is linear on $[x_k, (x_k + x_{k+1})/2]$, there exists a sufficiently small $\varepsilon > 0$ such that

$$|f(t) - s_\varepsilon(t)| \leqslant 1, \qquad t \in [x_k, x_k + \varepsilon].$$

Moreover, since $\|s_\varepsilon\| \to 0$ for $\varepsilon \to 0$, for sufficiently small $\varepsilon > 0$,

$$\|f - s_\varepsilon\|_{[x_k, x_{k+1}]} = 1$$

which implies that

$$\|f - s_\varepsilon\| = 1 = \|f - s_0\|.$$

This shows that $s_0 \neq s_\varepsilon \in P_{S_{m,k}}(f)$ and proves Theorem 2.

We note that the proofs of the above results show that the same statements hold, if we consider the mapping $\tilde{P}_{S_{m,k}} : C[a, b] \to 2^{S_{m,k} \cap C[a,b]}$, defined by $\tilde{P}_{S_{m,k}}(f) = P_{S_{m,k}}(f) \cap C[a, b]$ for all $f \in C[a, b]$, instead of $P_{S_{m,k}}$. It was shown by Schumaker [16] that $\tilde{P}_{S_{m,k}}(f) \neq \varnothing$ for all $f \in C[a, b]$. In [12] we incorrectly announced the result that $\tilde{P}_{S_{m,k}}$ is upper semicontinuous (compare the statement in Corollary 1 for $\tilde{P}_{S_{m,k}}$).

We finally consider a further continuity property. A continuous mapping $F: C[a, b] \to S_{m,k}$ is called a *continuous selection* for $P_{S_{m,k}}$ if $F(f) \in P_{S_{m,k}}(f)$ for all $f \in C[a, b]$.

In the fixed knot case, it was proved by Nürnberger and Sommer [13] that there exists a continuous selection for the metric projection $P_{S_m\binom{x_1, \ldots, x_k}{1, \ldots, 1}}$ if and only if $k \leqslant m+1$ (for further continuity results see Berens and Nürnberger [2], Nürnberger and Sommer [14], and Nürnberger [11]). On the other hand, the problem of the existence of continuous selections for $P_{S_{m,k}}$ is unsolved at present.

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640/71/2-3